The Fuel Tanker Problem – Tristan J. Newman

Introduction:

In space, the effective currency governing how far or fast a spacecraft can go is called **delta-v** (Δv), or change in velocity. In effect the total delta-v of a rocket is the measure of how much its speed would increase by expelling its entire mass of fuel through an engine. Of course, external forces such as gravity and atmospheric drag will cause the dissipation of kinetic energy from the rocket, so the actual delta-v required to achieve orbit from the surface of a body is always larger than the orbital velocity at the desired altitude.

Because of the first (energy can be neither created nor destroyed) and second (there is no process which can completely convert energy from one form to another without dissipation losses) laws of thermodynamics, delta-v requires the expulsion of fuel through a physical engine. A physical engine has an intrinsic property called **specific impulse** (I_{SP}) that is the measure of how much thrust can be generated per unit weight of fuel burned per unit time. I_{SP} itself has units of time, but it is incorrect to think of it as a measure of time. However, when multiplied by an acceleration (typically the acceleration due to gravity at the surface of the Earth, or 9.8 meters per second per second), one gets a value commonly referred to as **exhaust velocity** (v_e), the effective velocity of the exhaust gases that pass through the nozzle of the physical engine.

The **Tsiolkovsky Rocket Equation** (TRE) uses the exhaust velocity of a physical engine to calculate the amount of delta-v attainable by a rocket during a maneuver (or series of maneuvers) given the mass of the rocket both before (m_0) and afterwards (m_f):

(TRE) $$\Delta v = v_e \ln\left(\frac{m_0}{m_f}\right)$$

TRE will become important later when discussing mass ratios. For now, some terms must be defined before we can comprehensively discuss the titular Fuel Tanker Equations.

Definitions:

- **Fuel Tanker** – A fuel tanker in this context is any spacecraft which is capable of making a round trip from a refueling point to a cargo loading point, where it will take on a certain pre-measured quantity of fuel and bring it back to the refueling point. A single stage and single engine (or configuration of engines) is used for the entire mission cycle.
- **Refueling Point** – The refueling point is the point in space at which the fuel tanker offloads its cargo, and replenishes its own tanks of burnable fuel.
- **Cargo Loading Point** – The point at which non-burnable fuel (cargo fuel) is loaded onto the tanker. No refueling takes place at the cargo loading point.
- **Mission Cycle** – A mission cycle is a round trip from the refueling point to the cargo loading point and back again.

- **Burnable Fuel** – Burnable fuel is the fuel that the tanker is allowed to burn during its designated mission cycle.
- **Non-Burnable (Cargo) Fuel** – Non-burnable fuel (cargo fuel) is fuel that the spacecraft is not allowed to burn during its designated mission cycle. A certain amount of cargo fuel is designated to become burnable fuel at the refueling point.

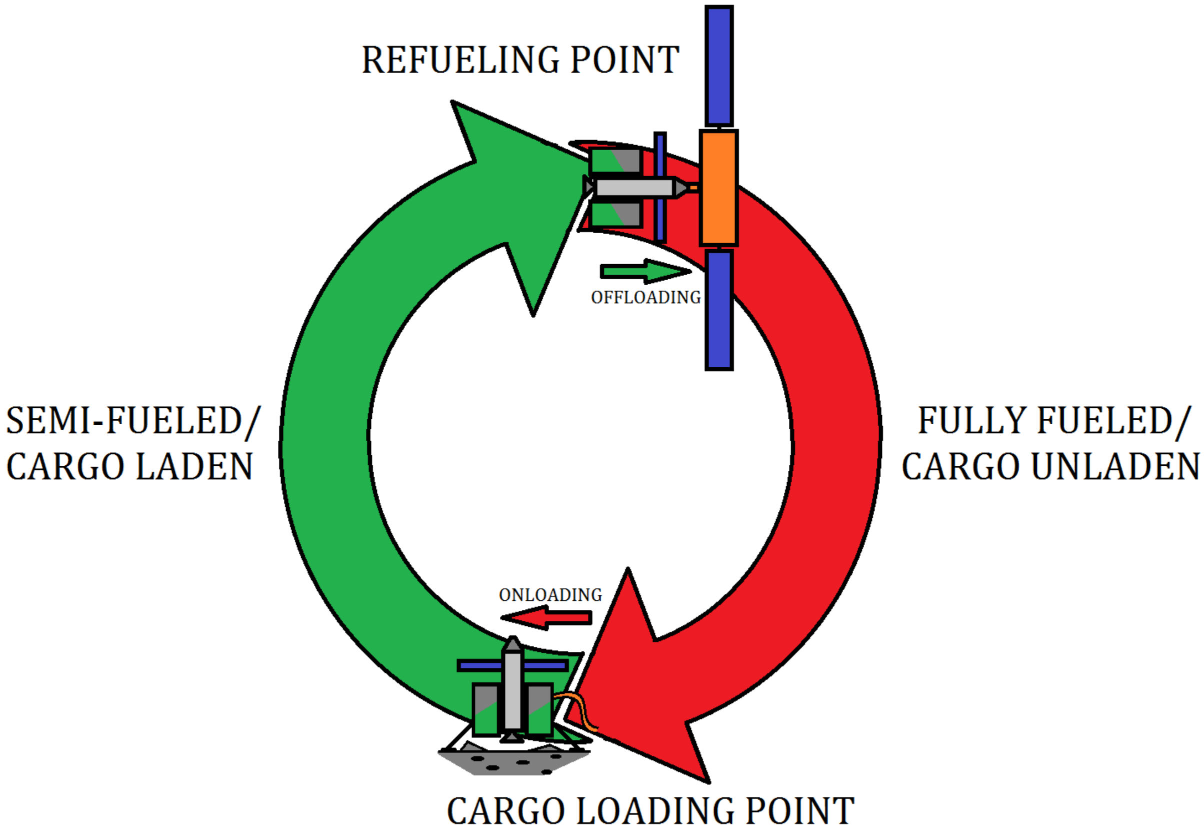

Figure 1 - Typical mission cycle for a fuel tanker type spacecraft.

Mission Constraints

There are two primary mission constraints and for a fuel tanker type spacecraft.

1. It must be able to complete a mission cycle on one engine or set of engines configured in a single stage. This is the reusability requirement, as a tanker should be useful for multiple mission cycles without need of replacement.

   Furthermore, each leg of the mission cycle must have consistent delta-v requirements – the amount of burnable fuel allowed is determined by the delta-v requirements for a given trip, and requiring the fuel tanker to expend any more or less delta-v may disturb the equilibrium of delivering and reusing cargo fuel for each mission cycle.

Quantitatively, this implies that the specific impulse is constant and the same for both legs of the mission cycle. If I_{SP} is constant, and the delta-v requirements for each leg is known then it follows from TRE that $R = \exp(-\Delta v/v_e)$, where R is the mass ratio m_f/m_0 of each leg of the mission cycle.

Therefore, if m_0 and m_1 are defined to be the tanker's mass at the start and end of the first leg respectively, and m_2 and m_3 are the tanker's mass at the start and end of the second leg respectively, then the following ratios must be constant.

(1) $$R_1 \equiv \frac{m_1}{m_0} = \exp\left(-\frac{\Delta v_1}{v_e}\right)$$

(2) $$R_2 \equiv \frac{m_3}{m_2} = \exp\left(-\frac{\Delta v_2}{v_e}\right)$$

2. The tanker should be designed such that it burns through all of the burnable fuel in its stores before reaching the refueling point. The only fuel that is not burned is the cargo fuel. Then, after refueling it must be able to offload a predetermined amount of fuel, using the remainder to refill its own burnable fuel storage tanks.

   Taking this requirement into consideration, the amount of burnable fuel can be expressed in terms of the total amount of cargo fuel to be carried, and the amount of cargo fuel that is to be offloaded at the refueling point.

(3) $$\Delta m_1 + \Delta m_2 = m_{cargo} - m_{offload}$$

   Where Δm_1 and Δm_2 are the respective amounts of fuel burned during the initial and final legs of the mission cycle.

Mass Targets for Mission Cycle Completion

As mentioned in the introduction, delta-v can only be achieved through reduction in mass. TRE shows that the primary driver for delta-v is mass difference: a high initial mass and a low final mass yield a larger delta-v. However, using more efficient engines (i.e. higher I_{SP}) will yield greater delta-v for a given mass of fuel. It was shown in equations (1) and (2) that the mass ratio of the tanker at the end of the mission cycle leg to that at the start can be expressed entirely as a function of exhaust velocity (which is directly proportional to I_{SP}), and the desired amount of delta-v to be achieved for that leg.

Since the ratios are constant known values, this means that the masses at each point of the mission cycle must also be constant known values. When engineering the spacecraft, these are known as **mass targets** for the mission.

The first mass target is the initial mass of the tanker when unladen with cargo but fully stocked in burnable fuel. To derive this, we must consider the definition set forth for the mass ratio of the initial leg

$$\mathrm{R}_1 \equiv \frac{m_1}{m_0} \Longrightarrow m_1 = \mathrm{R}_1 m_0$$

Between the point when the spacecraft had mass m_0 to when it had mass m_1, the amount of fuel burned had a mass of Δm_1. Therefore m_1 can be written as

$$m_1 = m_0 - \Delta m_1 = \mathrm{R}_1 m_0$$

(4) $$\Delta m_1 = (1 - \mathrm{R}_1) m_0$$

By applying the same algebra to the second leg, it follows that the second mass change is

(5) $$\Delta m_2 = (1 - \mathrm{R}_2) m_2$$

Summing equations (4) and (5) yields the left hand side of equation (3), therefore

$$(1 - \mathrm{R}_1) m_0 + (1 - \mathrm{R}_2) m_2 = m_{cargo} - m_{offload}$$

Since the second mission constraint requires that no burnable fuel be loaded back onto the spacecraft at the cargo loading point, the only mass change that occurs between m_1 and m_2 is the onloading of m_{cargo} in cargo fuel (i.e. $m_2 = m_1 + m_{cargo}$). Therefore equation (3) becomes

$$(1 - \mathrm{R}_1) m_0 + (1 - \mathrm{R}_2) m_1 + (1 - \mathrm{R}_2) m_{cargo} = m_{cargo} - m_{offload}$$

$$(1 - \mathrm{R}_1) m_0 + (1 - \mathrm{R}_2) \mathrm{R}_1 m_0 + (1 - \mathrm{R}_2) m_{cargo} = m_{cargo} - m_{offload}$$

All that remains is to solve algebraically for m_0.

$$[(1 - \mathrm{R}_1) + (1 - \mathrm{R}_2)\mathrm{R}_1] m_0 = [1 - (1 - \mathrm{R}_2)] m_{cargo} - m_{offload}$$

$$[1 - \mathrm{R}_1 + \mathrm{R}_1 - \mathrm{R}_1\mathrm{R}_2] m_0 = [1 - 1 + \mathrm{R}_2] m_{cargo} - m_{offload}$$

$$[1 - \mathrm{R}_1\mathrm{R}_2] m_0 = \mathrm{R}_2 m_{cargo} - m_{offload}$$

$$m_0 = \frac{1}{1 - \mathrm{R}_1\mathrm{R}_2}\left[\mathrm{R}_2 m_{cargo} - m_{offload}\right]$$

Note that the quantity $\mathrm{R}_1\mathrm{R}_2 = \exp(-\Delta v_1/v_e) \cdot \exp(-\Delta v_2/v_e) = \exp(-(\Delta v_1 + \Delta v_2)/v_e) \equiv R_{12}$ can be interpreted physically as the mass ratio over the course of the entire mission cycle if the fuel tanker does not take on cargo.

(6) $$\boxed{\boldsymbol{m_0 = \frac{1}{1-\mathrm{R}_{12}}\left[\mathrm{R}_2 m_{cargo} - m_{offload}\right]}}$$

Because $m_1 = \mathrm{R}_1 m_0$, the second mass target is

(7) $$\boxed{\boldsymbol{m_1 = \frac{R_1}{1-\mathrm{R}_{12}}\left[\mathrm{R}_2 m_{cargo} - m_{offload}\right]}}$$

Furthermore, since $m_2 = m_1 + m_{cargo}$, the third mass target can be expressed as

$$m_2 = \frac{R_1}{1-\mathrm{R}_{12}}\left[\mathrm{R}_2 m_{cargo} - m_{offload}\right] + m_{cargo}$$

Which can be simplified algebraically to

$$m_2 = \frac{1}{1-\mathrm{R}_{12}}\left[R_1\left(\mathrm{R}_2 m_{cargo} - m_{offload}\right) + (1-\mathrm{R}_{12})m_{cargo}\right]$$

$$m_2 = \frac{1}{1-\mathrm{R}_{12}}\left[R_{12} m_{cargo} - R_1 m_{offload} + m_{cargo} - \mathrm{R}_{12} m_{cargo}\right]$$

(8) $$\boxed{\boldsymbol{m_2 = \frac{1}{1-\mathrm{R}_{12}}\left[m_{cargo} - R_1 m_{offload}\right]}}$$

Finally, the mass ratio R_2 can be used to find the fourth and final mass target

(9) $$\boxed{\boldsymbol{m_3 = \frac{R_2}{1-\mathrm{R}_{12}}\left[m_{cargo} - R_1 m_{offload}\right]}}$$

To summarize, the four mass targets for mission cycle completion are:

$$\boxed{\begin{cases} \boldsymbol{m_0 = \dfrac{1}{1-\mathrm{R}_{12}}\left[\mathrm{R}_2 m_{cargo} - m_{offload}\right]} \\ \boldsymbol{m_1 = \dfrac{R_1}{1-\mathrm{R}_{12}}\left[\mathrm{R}_2 m_{cargo} - m_{offload}\right]} \\ \boldsymbol{m_2 = \dfrac{1}{1-\mathrm{R}_{12}}\left[m_{cargo} - R_1 m_{offload}\right]} \\ \boldsymbol{m_3 = \dfrac{R_2}{1-\mathrm{R}_{12}}\left[m_{cargo} - R_1 m_{offload}\right]} \end{cases}}$$

The Cargo-Offload Relation

Although useful in designing a spacecraft, the mass targets by themselves do not provide concrete values without knowing what to input for m_{cargo} and $m_{offload}$. These are as much properties of the spacecraft as the mass targets themselves are. The exact values of these depend on the mass of the rest of the spacecraft, which includes all structure and fuel tanks.

Structure mass is represented solely by the value m_{struct} which includes the aggregate mass of all non-fuel carrying components such as command/control units, RCS fuel, engine mass, decouplers, connections, and landing legs.

The mass of fuel carrying elements can be regarded with two separate values: those for burnable fuel carrying elements, and those for cargo carrying elements. This mass is a linear function of the mass of fuel carried. For burnable fuel:

$$m_{bft} = f_{bft} \cdot m_{fuel}$$

$$m_{bft} = f_{bft}(\Delta m_1 + \Delta m_2)$$

(10) $$m_{bft} = f_{bft}\left(m_{cargo} - m_{offload}\right)$$

Where $f_{bft} \equiv m_{bft}/m_{fuel}$ is the **burnable fuel tank factor**, or the total mass of fuel tanks which must be added to accommodate one unit mass of burnable fuel. There also exists a similarly-defined **cargo fuel tank factor** given by $f_{bft} \equiv m_{cft}/m_{cargo}$.

(11) $$m_{cft} = f_{cft} \cdot m_{cargo}$$

The initial fully fueled, cargo unladen mass m_0 can therefore be expressed in terms of the individual masses for the fundamental components of the spacecraft: Structure, burnable fuel tanks, and cargo fuel tanks (plus an additional term for the total mass of fuel carried).

(12) $$m_0 = m_{struct} + m_{bft} + m_{cft} + m_{fuel}$$

Substituting equations (10) and (11) into (12) yields

(13) $$m_0 = m_{struct} + f_{bft}\left(m_{cargo} - m_{offload}\right) + f_{cft} \cdot m_{cargo} + m_{cargo} - m_{offload}$$

Because equation (13) is equal to (6)

$$m_{struct} + f_{bft}\left(m_{cargo} - m_{offload}\right) + f_{cft} \cdot m_{cargo} + m_{cargo} - m_{offload} = \frac{1}{1 - \mathrm{R}_{12}}\left[\mathrm{R}_2 m_{cargo} - m_{offload}\right]$$

Unifying each term on basis of mass yields

$$m_{struct} + \left(1 + f_{bft} + f_{cft}\right)m_{cargo} + \left(-1 - f_{bft}\right)m_{offload} = \frac{\mathrm{R}_2}{1 - \mathrm{R}_{12}}m_{cargo} - \frac{1}{1 - \mathrm{R}_{12}}m_{offload}$$

$$m_{struct} + \left[1 + f_{bft} + f_{cft} - \frac{\mathrm{R}_2}{1 - \mathrm{R}_{12}}\right]m_{cargo} + \left[\frac{1}{1 - \mathrm{R}_{12}} - 1 - f_{bft}\right]m_{offload} = 0$$

(14) $$\boxed{\boldsymbol{m_{struct} + \left[1 + f_{bft} + f_{cft} - \frac{\mathrm{R}_2}{1-\mathrm{R}_{12}}\right]m_{cargo} + \left[\frac{\mathrm{R}_{12}}{1-\mathrm{R}_{12}} - f_{bft}\right]m_{offload} = 0}}$$

Equation (14) is called the Cargo-Offload Relation, as the total mass distribution of the tanker must satisfy it in order to be effective. Generally, either the cargo mass or the offload mass is known. For example, if a fuel tanker is designed to remove all the fuel from a surface base in one mission cycle then m_{cargo} is known. However, if the mission plan calls for a tanker to completely refill the stores of an orbiting space station then $m_{offload}$ would be the fixed quantity.

As an exercise, it can be shown that similarly expressing the tanker's mass at the end of the mission cycle (m_3) in terms of its structural components, and then equating that with equation (9) will also yield equation (14).

Profitability Analysis and the Positive Performance Condition

Recall that the second law of thermodynamics dictates that no process can perfectly convert one form of energy into another without some losses. This is particularly true in the case of a fuel tanker spacecraft. When discussing profitability in this context, the reference is to the mass of cargo fuel offloaded at the refueling point compared to the total mass of cargo fuel carried there.

(15) $$\text{Profitability} = P \equiv \frac{m_{offload}}{m_{cargo}}$$

With this in mind, consider again equation (14). The terms m_{struct}, f_{bft} and f_{cft} are called **design constraints**, and are intrinsic properties of a physical tanker. Ideally they are as low as possible, but they cannot ever be zero. However, setting them equal to zero and analyzing the simplified equation provides a useful set of bounds for m_{cargo} and $m_{offload}$. In such a case, equation (14) becomes

$$\left[1 - \frac{\mathrm{R}_2}{1 - \mathrm{R}_{12}}\right] m_{cargo} + \left[\frac{\mathrm{R}_{12}}{1 - \mathrm{R}_{12}}\right] m_{offload} = 0$$

$$[1 - \mathrm{R}_{12} - \mathrm{R}_2] m_{cargo} + \mathrm{R}_{12} \cdot m_{offload} = 0$$

With the constraint set forth in equation (15), this becomes

$$[1 - \mathrm{R}_{12} - \mathrm{R}_2] m_{cargo} + \mathrm{R}_{12} P \cdot m_{cargo} = 0$$

$$[1 - \mathrm{R}_{12} + \mathrm{R}_{12} P - \mathrm{R}_2] m_{cargo} = 0$$

(16) $$[1 - (1 - P)\mathrm{R}_{12} - \mathrm{R}_2] m_{cargo} = 0$$

Where (1-P) is a positive quantity, since P must be less than one due to the second law of thermodynamics. Furthermore, since m_{cargo} is a positive known quantity, the leading coefficient must be zero.

$$1 - (1 - P)\mathrm{R}_{12} - \mathrm{R}_2 = 0$$

$$(1 - P)\mathrm{R}_{12} + \mathrm{R}_2 = 1$$

$$1 - P = \frac{1 - \mathrm{R}_2}{\mathrm{R}_{12}}$$

$$P = 1 - \frac{1 - \mathrm{R}_2}{\mathrm{R}_{12}} = \frac{\mathrm{R}_{12} + \mathrm{R}_2 - 1}{\mathrm{R}_{12}}$$

In this instance it is more convenient to express P in terms of the mass ratios for each leg of the mission cycle.

(17) $$P_{max} = \frac{(1+R_1)R_2 - 1}{R_1 R_2}$$

Equation (17) is the **maximum profitability** that can be attained by a fuel tanker. Because this assumes the spacecraft (excluding the fuel) is entirely made of massless parts, it is impossible to build a spacecraft with the same delta-v requirements, and cargo load capacity that has a profitability exceeding this value.

It can be simply shown that $P_{max} < 1$ for any combination of values for R_1 and R_2, which is in holding with the first law of thermodynamics – energy can neither be created nor destroyed. Since $P_{max} = 1$ would imply that no fuel was burned during the mission cycle, and that kinetic energy was created from nothing.

Conversely, equation (17) has no built-in minimum. From a practical standpoint however, when $P_{max} \leq 0$ the spacecraft has entered a **design-based profitability deficit**. Physically this means that based on the current delta-v requirements, a fuel tanker designed to carry a certain amount of fuel would burn more fuel than it would deliver.

It is possible to calculate the relationship between the values of R_1 and R_2 that would yield a deficit by setting equation (17) less than or equal to zero.

$$P_{max} = \frac{(1+R_1)R_2 - 1}{R_1 R_2} \leq 0$$

$$(1+R_1)R_2 - 1 \leq 0$$

$$1 + R_1 \leq \frac{1}{R_2}$$

$$R_1 \leq \frac{1 - R_2}{R_2}$$

In order to avoid a deficit, the tanker design must satisfy the inverse:

(18) $$R_1 > \frac{1 - R_2}{R_2}$$

Inequality (18) is called the **positive performance condition** for an ideal fuel tanker. The values of R_1 and R_2 that satisfy it represent the mission cycle delta-v's for which the spacecraft will deliver a positive profitability rating. (18) provides a useful upper bound for a design's delta-v requirements.

However, in reality the weight of spacecraft components is not negligible. The **propellant mass fraction** of a spacecraft – the total mass of fuel in a spacecraft compared to its initial mass – is an important determiner of how efficient a spacecraft is. As a point of reference, the Apollo Lunar Module had a PMF of approximately 0.7, and its ascent stage had a PMF as low as 0.5 [http://www.braeunig.us/space/specs/lm.htm].

Condition (15) can be applied to equation (14), and after some algebra which is not in itself particularly insightful, will yield an expression for the profitability of a physical fuel tanker:

(19)
$$\boxed{P_{max} = \frac{R_2 - (f_{ft} + \varepsilon_{struct})(1 - R_{12})}{R_{12} - (1 - R_{12}) f_{bft}}}$$

Where f_{ft} is called the **fuel tank factor** and is defined by $f_{ft} \equiv 1 + f_{bft} + f_{cft}$ and can be physically interpreted as the factor by which a given quantity of fuel increases the total mass of the tanker. ε_{struct} is called the **structural inefficiency** of the design given by $\varepsilon_{struct} \equiv m_{struct}/m_{cargo}$. For an ideal tanker, $f_{ft} = 1$ and $\varepsilon_{struct} = 0$.

Note that equation (19) simplifies to equation (17) for an ideal tanker. The terms that represent the mass contributions of structural components have a damping effect on the end profitability. Intuitively, this would mean that as a heavier spacecraft carrying the same amount of fuel will in fact burn more of it to achieve a given delta-v.

As before, equation (19) can be used to determine the positive performance condition. This time the result is generally applicable to a physical fuel tanker:

(20)
$$\boxed{R_1 > \frac{(f_{ft} + \varepsilon_{struct}) - R_2}{(f_{ft} + \varepsilon_{struct}) R_2}}$$

Like equation (19), (20) simplifies to the ideal model (18) when the mass of structural components is neglected.

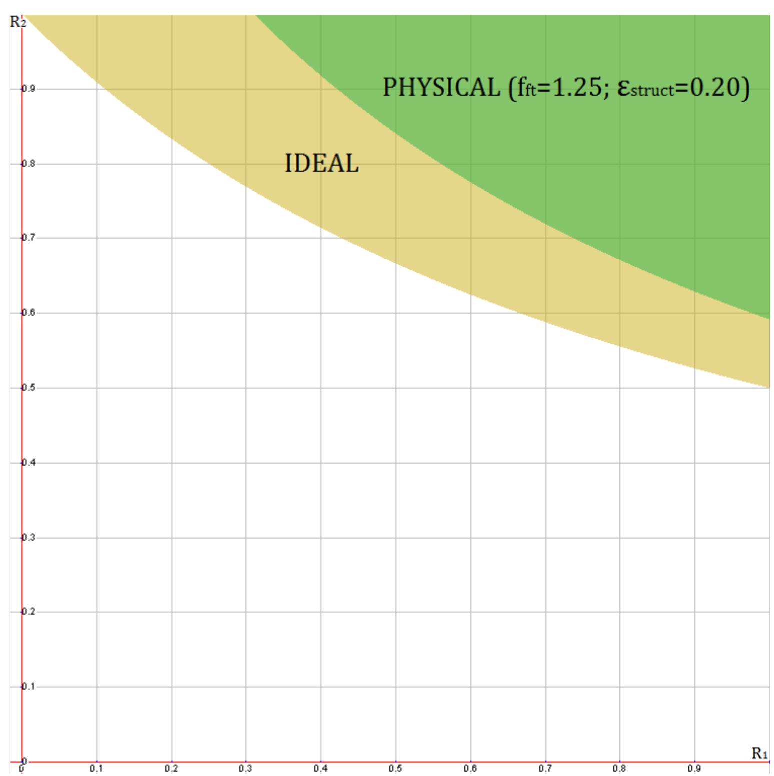

Figure 2 - Example positive performance regimes for an ideal and physical fuel tanker. [http://www.gcalc.net]

Parting Observations

The purpose of an orbital fuel tanker is to move a quantity of fuel from one location and supply it to another. It has been shown here that doing so will never result in the full amount being transferred, while still maintaining the reusability of the tanker. For larger quantities of fuel being moved by cheaply constructed disposable transports, the profitability is greater over the short term. However, to conduct multiple return trips reusability is the better investment – for a well-designed fuel tanker, the profitability will eventually overcome the costs associated with its construction and launch.

Although the equations outlined here do not concretely outline exactly how such a tanker could be built, they do provide the means to determine whether a prospective design has the capability to fulfill its mission requirements and objectives.

If the delta-v requirements are known then the profitability equation, positive performance condition, and cargo-offload relation will help to determine the cargo and offload ratings for a particular design. Then, the mass target equations can be used to guide the construction of the spacecraft on a by-weight basis.

Together, these equations will help designers reduce the number of iterations designers will need to undergo before design functionality convergence is achieved.